MATH 221: Midterm 1

Name: ______________________________

Directions: No technology, internet, or notes. **Simplify all expression for full credit**. If you have a question, ask me. Good luck!

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
		50

1. Short answer questions:

   (a) True or false: We can simplify $\dfrac{x^2+x-2}{x-1}$ by crossing out the x's to become $\dfrac{x^2-2}{-1}$. If not, properly simplify the expression.

   False; you can only cancel if $+$'s and $-$'s are encapsulated in parentheses.

   The proper simplification is

$$\frac{x^2+x-2}{x-1} = \frac{(x+2)(x-1)}{(x-1)} = x+2$$

   (b) Suppose $f(x)$ is a continuous function with a domain of $\mathbb{R}$. We know that $f(-8) = -8$ and $f(8) = -8$. Must $f(x)$ have an x-intercept in the interval $(-8, 8)$?

   No. You can connect $(-8, -8)$ and $(8, -8)$ with a horizontal line. The associated function is $f(x) = -8$ and is continuous everywhere but it does not cross the x-axis.

   (c) If $f(x) = \dfrac{x}{1-x}$, find $f(x^2-1)$.

   Replace all the x's in $f(x) = \dfrac{x}{1-x}$ with x^2-1. We have

$$f(x^2-1) = \frac{x^2-1}{1-(x^2-1)} = \frac{x^2-1}{1-x^2+1} = \frac{x^2-1}{2-x^2}$$

   (d) True or false: The function

$$f(x) = \frac{x^5+x^4-x^3+x^2+1}{x^2+1}$$

   is continuous on $\mathbb{R}$.

   True. $f(x)$ is a rational function. Therefore, we check where $x^2+1=0$ and remove those x values. But we have

$$\begin{aligned} x^2+1 &= 0 \\ x^2 &= -1 \\ x &= \pm\sqrt{-1} = \pm i \notin \mathbb{R} \end{aligned}$$

   so we do not need to exclude any real numbers. Thus $f(x)$ is continuous on $\mathbb{R}$.

2. Suppose

$$f(x) = \begin{cases} (x-1)^2 + 2 & x \geq 1 \\ \dfrac{x^2-1}{x-1} & x < 1 \end{cases}$$

Find $\lim\limits_{x \to 1} f(x)$ **using left and right hand limits**.

We will use one sided limits.

Left

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x^2-1}{x-1} = \frac{1^2-1}{1-1} = \frac{0}{0}$$

thus we have an indeterminate form. Let's cancel common factors. We have

$$\lim_{x \to 1^-} \frac{x^2-1}{x-1} = \lim_{x \to 1^-} \frac{(x+1)(x-1)}{x-1} = \lim_{x \to 1^-} x+1 = 1+1 = 2$$

Right

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-1)^2 + 2 = (1-1)^2 + 2 = 2$$

Because $\lim_{x \to 1^-} f(x) = 2 = \lim_{x \to 1^+} f(x)$, we can conclude $\lim_{x \to 1} f(x) = 2$.

3. Suppose

$$f(x) = \begin{cases} x+5 & x<0 \\ 2 & x=0 \\ -x^2+5 & x>0 \end{cases}$$

Find where $f(x)$ is continuous **using the definition of continuity**.

This piecewise function is continuous when $x < 0$ and $x > 0$ because both $x+5$ and $-x^2+5$ are polynomials. Thus we only need to check $x = 0$ for potential issues. Using the definition of continuity at $x = 0$:

(a) $f(0)$ is defined and we have $f(0) = 2$.

(b) $lim_{x\to 0} f(x)$ is defined because

  i. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} -x^2+5 = 0+5 = 5$

  ii. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x+5 = 0+5 = 5$

  so we know $\lim_{x\to 0} f(x) = 5$.

(c) However,

$$2 = f(0) \neq \lim_{x\to 0} f(x) = 5$$

  so $f(x)$ is not continuous at $x = 0$ because it violates condition three of the definition.

Therefore, $f(x)$ is continuous on $(-\infty, 0) \cup (0, \infty)$.

4. Suppose

$$f(x) = x^2 - x$$

Find $f'(x)$ using the limit definition of the derivative.

Using the definition of derivative:

$$\begin{aligned}
f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^2 - (x+h) - (x^2 - x)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x - h - x^2 + x}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 - h}{h} \\
&= \lim_{h\to 0} \frac{h(2x + h - 1)}{h} \\
&= \lim_{h\to 0} 2x + h - 1 \\
&= \lim_{h\to 0} 2x + \lim_{h\to 0} h - \lim_{h\to 0} 1 \\
&= 2x + 0 - 1 \\
&= 2x - 1
\end{aligned}$$

5. Find

$$\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t}$$

Using limit laws, we see

$$\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} = \frac{\sqrt{1+\lim_{t\to 0} t}-\sqrt{1-\lim_{t\to 0} t}}{\lim_{t\to 0} t} = \frac{\sqrt{1+0}-\sqrt{1-0}}{0} = \frac{0}{0}$$

This is an indeterminate form, so we multiply by the conjugate radical:

$$\begin{aligned}
\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} &= \lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} \cdot \frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}} \\
&= \lim_{t\to 0} \frac{(\sqrt{1+t})^2-(\sqrt{1-t})^2}{t(\sqrt{1+t}+\sqrt{1-t})} \\
&= \lim_{t\to 0} \frac{1+t-(1-t)}{t(\sqrt{1+t}+\sqrt{1-t})} \\
&= \lim_{t\to 0} \frac{1+t-1+t}{t(\sqrt{1+t}+\sqrt{1-t})} \\
&= \lim_{t\to 0} \frac{2t}{t(\sqrt{1+t}+\sqrt{1-t})} \\
&= \lim_{t\to 0} \frac{2}{\sqrt{1+t}+\sqrt{1-t}} \\
&= \frac{2}{\sqrt{1+\lim_{t\to 0} t}+\sqrt{1-\lim_{t\to 0} t}} \\
&= \frac{2}{\sqrt{1+0}+\sqrt{1-0}} \\
&= \frac{2}{2} \\
&= 1
\end{aligned}$$