MATH 221: Midterm 1

Name: ______________________________

Directions: No technology, internet, or notes. **Simplify all expressions for full credit**. If you have a question, ask me. Good luck!

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
		50

1. Short answer questions:

   (a) True or false: We can simplify $\dfrac{3x^5 + 2x - 2}{2x - 1}$ by crossing out the $2x$ to become $\dfrac{3x^5 - 2}{-1}$. If not, properly simplify the expression.

   False, the expression is already simplified.

   (b) If $\lim_{x \to 1^+} f(x) = 2$ and $\lim_{x \to 1^-} f(x) = 2.000000001$, then it is true that $\lim_{x \to 1} f(x) = 2$.

   False, the left-hand limit and the right-hand limit do not agree so $\lim_{x \to 1} f(x)$ cannot exist.

   (c) If $f(x) = \dfrac{x}{1 - x}$, find $f(x^2 - 1)$.

   Replace all the x's in $f(x) = \dfrac{x}{1 - x}$ with $x^2 - 1$. We have

$$f(x^2 - 1) = \frac{x^2 - 1}{1 - (x^2 - 1)} = \frac{x^2 - 1}{1 - x^2 + 1} = \frac{x^2 - 1}{2 - x^2}$$

   (d) True or false: The function

$$f(x) = \frac{x^5 + x^4 - x^3 + x^2 + 1}{x^2 - 1}$$

   is continuous on $\mathbb{R}$.

   False. We only need to exclude x-values where $x^2 - 1 = 0$:

$$\begin{aligned} x^2 - 1 &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

   So $f(x)$ is continuous on $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

2. Find

$$\lim_{t\to 0} \frac{t}{\sqrt{1+t}-\sqrt{1-t}}$$

Using limit laws, we see

$$\lim_{t\to 0} \frac{t}{\sqrt{1+t}-\sqrt{1-t}} = \frac{\lim_{t\to 0} t}{\sqrt{1+\lim_{t\to 0} t}-\sqrt{1-\lim_{t\to 0} t}} = \frac{0}{\sqrt{1+0}-\sqrt{1-0}} = \frac{0}{0}$$

This is an indeterminate form, so we multiply by the conjugate radical:

$$\begin{aligned}
\lim_{t\to 0} \frac{t}{\sqrt{1+t}-\sqrt{1-t}} &= \lim_{t\to 0} \frac{t}{\sqrt{1+t}-\sqrt{1-t}} \cdot \frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}} \\
&= \lim_{t\to 0} \frac{t(\sqrt{1+t}+\sqrt{1-t})}{(\sqrt{1+t})^2-(\sqrt{1-t})^2} \\
&= \lim_{t\to 0} \frac{t(\sqrt{1+t}+\sqrt{1-t})}{1+t-(1-t)} \\
&= \lim_{t\to 0} \frac{t(\sqrt{1+t}+\sqrt{1-t})}{1+t-1+t} \\
&= \lim_{t\to 0} \frac{t(\sqrt{1+t}+\sqrt{1-t})}{2t} \\
&= \lim_{t\to 0} \frac{\sqrt{1+t}+\sqrt{1-t}}{2} \\
&= \frac{\sqrt{1+\lim_{t\to 0} t}+\sqrt{1-\lim_{t\to 0} t}}{2} \\
&= \frac{\sqrt{1+0}+\sqrt{1-0}}{2} \\
&= \frac{2}{2} \\
&= 1
\end{aligned}$$

3. Suppose

$$f(x) = \begin{cases} x+5 & x<0 \\ 2 & x=0 \\ -x^2+5 & x>0 \end{cases}$$

Find where $f(x)$ is continuous **using the definition of continuity**.

This piecewise function is continuous when $x < 0$ and $x > 0$ because both $x+5$ and $-x^2+5$ are polynomials. Thus we only need to check $x = 0$ for potential issues. Using the definition of continuity at $x = 0$:

(a) $f(0)$ is defined and we have $f(0) = 2$.

(b) $lim_{x\to 0} f(x)$ is defined because

i. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} -x^2 + 5 = 0 + 5 = 5$

ii. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x + 5 = 0 + 5 = 5$

so we know $\lim_{x\to 0} f(x) = 5$.

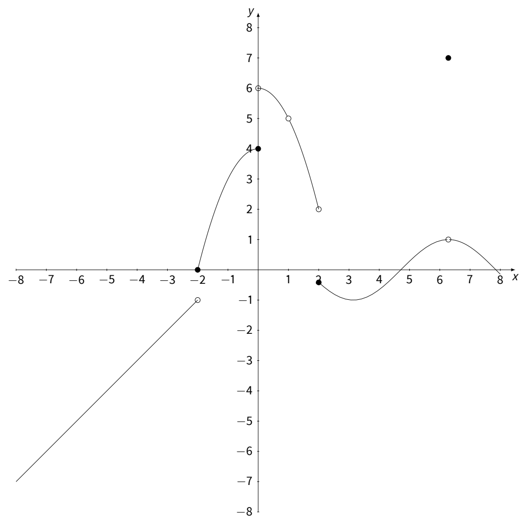
(c) However,

$$2 = f(0) \neq \lim_{x\to 0} f(x) = 5$$

so $f(x)$ is not continuous at $x = 0$ because it violates condition three of the definition.

Therefore, $f(x)$ is continuous on $(-\infty, 0) \cup (0, \infty)$.

4. Suppose a function has the following graph



Find all the x-values where $f(x)$ is discontinuous. For each one, state the exact reason **from the definition of continuity** why it is discontinuous.

Using the definition of continuity, $f(x)$ is discontinuous at:

(a) $x = -2$ because $\lim_{x \to -2^-} f(x) = -1 \neq 0 = \lim_{x \to -2^+} f(x)$ so $\lim_{x \to -2} f(x)$ does not exist, violating condition two in the definition.

(b) $x = 0$ because $\lim_{x \to 0^-} f(x) = 4 \neq 6 = \lim_{x \to 0^+} f(x)$ so $\lim_{x \to 0} f(x)$ does not exist, violating condition two in the definition.

(c) $x = 1$ because $f(1)$ is not defined, violating condition 1 in the definition.

(d) $x = 2$ because $\lim_{x \to 2^-} f(x) = 2 \neq -0.4 = \lim_{x \to 2^+} f(x)$ so $\lim_{x \to 2} f(x)$ does not exist, violating condition two in the definition.

(e) $x = 6.28$ because $\lim_{x \to 6.28} f(x) = 1$ while $f(6.28) = 7$, so $\lim_{x \to 6.28} f(x) \neq f(6.28)$, violating condition three in the definition.

5. Suppose

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \geq 1 \\ (x-1)^2+3 & x < 1 \end{cases}$$

Find $\lim\limits_{x\to 1} f(x)$ **using left and right hand limits**.

We will use one sided limits.

Left

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2-4}{x-2} = \frac{1^2-4}{1-2} = \frac{-3}{-1} = 3$$

Right

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1)^2+3 = (1-1)^2+3 = 3$$

Because $\lim_{x\to 1^-} f(x) = 3 = \lim_{x\to 1^+} f(x)$, we can conclude $\lim_{x\to 1} f(x) = 3$.